Astro 250 – Super Earths – Problem Set 7

Questions to Ponder While Reading

Problem 1. Sonic Point for Isothermal Wind

(a) For an isothermal wind, $T$ is constant and there is no need to solve an energy equation. From equations (1) and (2) from Kasting and Pollack, prove that

$$\left(\mathcal{M} - \frac{1}{\mathcal{M}}\right) \frac{d\mathcal{M}}{dr} = \frac{2}{r} - \frac{GM}{r^2c_s^2}$$

(1)

where $\mathcal{M} \equiv u/c_s$ equals the Mach number of the flow and $c_s = (kT/m)^{1/2}$ is the sound speed in an isothermal gas.

(b) Thereby show that for an isothermal flow that accelerates from subsonic to supersonic speeds, the sonic point is attained at $r = GM/2c_s^2$.

Problem 2. Re-Casting the Energy Equation

(a) From first principles, the energy equation reads

$$\frac{\partial \epsilon}{\partial t} + \nabla \cdot (\epsilon \vec{u}) = -P \nabla \cdot \vec{u} - \nabla \cdot F_{\text{con}} + q$$

(2)

where $\epsilon$ is the internal energy per unit volume, $P$ is the gas pressure, $\vec{u}$ is the gas velocity, $F_{\text{con}}$ is the energy flux carried by conduction, and $q$ is the heating rate due to viscous dissipation and/or radiation.

For an ideal gas, $P = \rho kT/m$ and $\epsilon = P/(\gamma - 1)$.

Show that equation (3) of Kasting and Pollack follows from the first-principles energy equation.\(^1\) I had to use their mass conservation equation (1).

(b) Is equation (7) of Kasting and Pollack consistent with their equation (3)?

Problem 3. Fluxes

(a) From the textbook “Theory of Planetary Atmospheres” by Chamberlain & Hunten (1987), the flux of a trace constituent equals

\(^1\)I think their $q$ has the wrong sign.
\[ F_i = n_i u_i = -D \frac{\partial n_i}{\partial r} - D \frac{m_i g}{kT} n_i - Kn \frac{\partial}{\partial r} \left( \frac{n_i}{n} \right) \]  

(3)

where the notation is the same as in Kasting and Pollack.

Identify the physical meaning of each term in the above equation. Convince yourself that each term is correct.

(b) Show that equation (17) of Kasting and Pollack follows from the above equation.

(c) Estimate the order of magnitude of \(D\) yourself. Compare to equations (20)–(23) of Kasting and Pollack.